LATTICE HOMOMORPHISMS

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Abstract

Lattice homomorphism, a fundamental concept in abstract algebra , order theory and mathematichal analysis, serves as a vital bridge connecting different lattices within various mathematical and computational domains. This abstract explores the significance, properties, and applications of lattice homomorphisms. Lattice homomorphisms act as mathematical mappings, preserving the essential lattice structure while facilitating the study of relationships between lattices. By meticulously conserving the meet and join operations, they reveal hidden patterns, enabling translation of concepts from one lattice to another. In addition to their theoretical importance, lattice homomorphisms find practical applications in data structures, formal logic, and fields where the organization and comparison of elements are central. Understanding the intricacies of lattice homomorphisms enriches our comprehension of lattices and extends their relevance to diverse areas of mathematics and computer science.

Lattice Homomorphisms

In this thesis a special class of positive operators will be studied. These are the operators that preserve the lattice operations, and they are known as lattice (or Riesz) homomorphisms.

Definition

An operator $T : E \to F$ between two Riesz spaces is said to be a lattice (or Riesz) homomorphism whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in E$. Observe that every lattice homomorphism $T : E \to F$ is necessarily a positive operator. Indeed, if $x \in E^+$, then

$$T(x) = T(x \vee 0) = T(x) \vee T(0) = [T(x)] \geq 0$$

Theorem

Assume that an operator $E \to F$ between two Riesz spaces is $one-to-one$ and onto. Then T is a lattice isomorphism if and only if T and T^{-1} are both positive operators.

Proof.

If T is a lattice isomorphism, then clearly T and T^{-1} are both positive operators. For the converse assume that T and T^{-1} are both positive operators and that x, y ∈ E. *From* x≤ $x \vee y$ and $y \leq x \vee y$, it follows $T(x) \leq T(x \vee y)$ and $T(y) \leq T(x \vee y)$ holds, and so

$$T(x) \vee T(y) \leq T(x \vee y)$$

Similarly, we see that $T^{-1}(u) \vee T^{-1}(v) \vee T^{-1}(u \vee v)$ for all $u, v \in$ F.
For $u = T(x)$ and $v = T(y)$ the last inequality yields $x \vee y \leq T^{-1}(T(x) \vee T(y))$ and by applying T it follows that $T(x \vee y) \leq T(x) \vee T(y)$. This combined with the above equation implies $T(x \vee y) = T(x) \vee T(y)$ so that the operator T is a lattice homomorphism.
Our next goal is to investigate the relationship between lattice homomorphisms and interval preserving operators. An operator $T : E \to F$ between two Riesz spaces is said to be interval preserving whenever T is a positive operator and $T[0, x] = [0, Tx]$ holds for each $x \in$ E$^+$.

Note that the range of an interval preserving operator is an ideal. The converse of the latter is, of course, false. For instance, the positive operator

$T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, y)$, is onto, and hence its range is an ideal. However, $0 \leq (0, 1) \leq (2, 1) = T(1, 1)$ holds and there is no vector $0 \leq (x, y) \leq (1, 1)$ satisfying $T(x, y) = (0, 1)$.

Theorem

For a Riesz space E the following statements are equivalent.

1. E has the $\sigma - order$ continuity property.
2. Every lattice homomorphism from E to any Archimedean Riesz space is $\sigma - order$ continuous.
3. Every uniformly closed ideal of E is a $\sigma - ideal$.

Proof. (1) ⇒ (2) Every lattice homomorphism is a positive operator, and the desired conclusion follows.
(2) ⇒ (3) Let A be a uniformly closed ideal of E. Then the Riesz space E/A is Archimedean.Since the canoncial projection of E onto E/A is a lattice homomorphism, by our hypothesis it must be $\sigma - order$ continuous. Therefore its kernel, the ideal A, must be a $\sigma - ideal$.
(3) ⇒ (1) Let F be an Archimedean Riesz space, and let $T : E \to F$ be a positive operator. Since F is an order dense Riesz subspace of its Dedekind completion F^δ we see that $T : E \to F$ is $\sigma - order$ continuous if and only if $T : E \to F^\delta$ is $\sigma - order$ continuous.This means that we can assume without loss of generality that F is Dedekind complete.

Theorem

Let E and F be two Riesz spaces with F Dedekind complete. If G is a majorizing Riesz subspace of E and $T : G \to F$ is a lattice homomorphism, then T extends to all of E as a lattice homomorphism.

Proof. Let $T : G \to F$ be a lattice homomorphism. By using convex set $E(T)$ has extreme points; let S be such a point. Then, we have $inf\{S \mid x - y \mid : y \in G\} = 0$ for each $x \in E$.

Now fix $x \in E$. Then for each $y \in G$, the fact that $T : G \to F$ is a lattice homomorphism implies $S \mid y \mid = T \mid y \mid = \mid Ty \mid = \mid Sy \mid$, and so

$$S \mid x \mid \leq S \mid x - y \mid + S \mid = S \mid x - y \mid + \mid Sy \mid \leq S \mid x - y \mid + \mid Sy - Sx \mid + \mid Sx \mid \leq 2S \mid x - y \mid + \mid Sx \mid$$

holds for all $y \in G$. From this , taking the infimum , it follows that

$$S \mid x \mid \leq \mid Sx \mid \leq S \mid x \mid$$

holds for all $y \in E$. So , S is a lattice homomorphic extension of T.

Example

Let $E = C[0, 1]$, $F = \mathbb{R}$ and let $G = \{\lambda 1 : \lambda \in \mathbb{R}\}$, where 1 denotes the constant function one . Clearly , G is a Riesz subspace majorizing E. Define $T : G \to \mathbb{R}$ by $T(\lambda 1) = \lambda$, and note that T is a lattice homomorphism.
Using the Riesz representation theorem it is easy to see that convex set of all probablity measures on $[0, 1]$. In other words, $S \in E(T)$ is a lattice homomorphic extension of T if and only if there exist some $t \in [0, 1]$ such that

$$S(f) = f(t) \int f(t)\delta_t \, dt$$

holds for all $f \in C[0, 1]$.

References

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